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# CONTACT PROBLEM WITH SEMIUNKNOWN BOUNDARY OF THE CONTACT REGION

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A. B. KOVURA and V. I. MOSSAKOVSKII
(Dnepropetrovsk)

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The contact problem of frictionless impression of a stamp in an elastic half-space is considered. The lateral surface of the stamp is a circular cylinder, while its base is an elliptic paraboloid. The case when the force of the impressing stamp reaches a value such that part of the edge obtained by the intersection of the cylindrical surface and the elliptic paraboloidal surface is penetrating into the boundary of the elastic half-space is investigated. In this case a mixed problem of the theory of elasticity presents itself with a partially unknown domain of separation of the boundary conditions on the half-space surface.

Determination of the pressure under the base of the stamp by using integral transforms [1] and power series in a small parameter [2] reduces to the construction of trigonometric expansions of the function characterizing the deviation of the contact domain from a circle. An approximate method based on point-interpolation approximations of the relationships determining the desired function [3], and on the analysis of the rate of decrease of the trigonometric expansion coefficients is proposed for the solution of the problem obtained, which can be considered as an extension of a harmonic analysis problem.

The contact problem of elasticity theory with a partially known boundary of the contact zone was apparently considered first in [4], which was devoted to a study of the pressure of an oblique circular stamp on an elastic half-space. The structure of the solution is constructed in [5] and a numerical result is presented for the problem of frictionless impression of a circular stamp whose base is an elliptic paraboloid into a half-space. Solutions are obtained in [4, 5] on the basis of the Rvachev method of $R$-functions [6].

The last of the mentioned problems is solved below by using classical mathematical physics methods under the assumption that the elliptical paraboloid is almost circular.

1. In an elastic half-space $z \leqslant 0$ let a rigid stamp with the cylindrical side surface $\rho = R$ and base in the form of an elliptic paraboloid be impressed without friction ($z$, $\rho$, $\varphi$ are cylindrical coordinates)

$$z = (A + \alpha \cos 2\varphi)\, \rho^2$$
$$A = {}^1/_2\, (a^{-2} + b^{-2}), \quad \alpha = {}^1/_2\, (a^{-2} - b^{-2}), \quad b < a$$

The quantity $|\alpha|$ is considered small compared to $R^2$, i.e., the elliptic paraboloid is almost circular.

Depending on the magnitude of the force applied to the stamp, or equivalently, on the vertical displacement of the stamp $w_0$, three qualitatively different kinds of contact interaction are possible. Let us consider the intermediate case when only a part of the endface line of the stamp is contiguous to the elastic medium. The boundary of the contact domain consists of the circular arcs $a_1a_2$, $a_3a_4$ on which the pressures are unlimited, and the previously unknown sections $a_2a_3$, $a_4a_1$ which are zero pressure lines (Fig. 1). The location of the points $a_j$ $(j = 1, 2, 3, 4)$ near which the change in the nature of the contact pressures occurs depends on the magnitude of the vertical displacement of the stamp, and is governed by one parameter, the angle $\psi$, because of the symmetry of the contact zone.

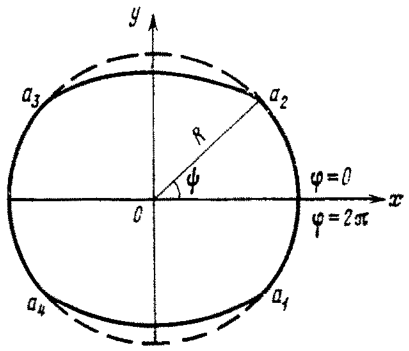


Fig. 1

2. As is known [7], to find the pressure under the base of the stamp, the function $p(\rho, \varphi) = u_z'(\rho, 0, \varphi)$ must be determined in the contact domain $S$ on the basis of the fact that the harmonic function $u(\rho, z, \varphi)$ in the half-space $z < 0$ should satisfy the boundary conditions

$$
\begin{aligned}
&u(\rho, 0, \varphi) = hw(\rho, \varphi) \quad \text{in} \quad S \\
&u_z'(\rho, 0, \varphi) = 0 \quad \text{outside} \quad S \\
&(h = {}^1/_2 E(1 - \nu^2)^{-1}, \quad w(\rho, \varphi) = z(\rho, \varphi) - w_0)
\end{aligned}
\tag{2.1}
$$

Here $w(\rho, \varphi)$ is the settlement under the stamp, $E$ is the elastic modulus, and $\nu$ is the Poisson's ratio.

In the case under consideration, the boundary of the domain $S$ is known only partially, hence the function $\rho(\varphi)$ in the equation of the boundary $\rho^2 = \rho^2(\varphi)$ is to be determined.

Following [1, 2], we reduce the boundary value problem (2.1) to a linear conjugate problem for the plane $\zeta = \xi + i\eta$ with the slit $|\xi| < \rho(\varphi)$

$$
\begin{aligned}
&\Phi^+ + \Phi^- = (4A\xi^2 - w_0 + {}^8/_3\alpha\xi^2 \cos 2\varphi) + G(\xi, \varphi) \\
&G(\xi, \varphi) = P_0(\xi)\cos 2\varphi + P_2(\xi)\cos 4\varphi + \ldots, \quad |\xi| < \rho(\varphi)
\end{aligned}
$$

where $P_{2n}(\xi)$ is a polynomial of order $2n$ with unknown coefficients, and $\varphi$ is a parameter.

We represent the function $\rho^2(\varphi)$ and the function $G(\zeta, \varphi)$, and $\Phi(\zeta, \varphi)$ which are analytic in the variable $\zeta$, in the form of expansions in a small parameter ([2, 8]

$$
\rho^2(\varphi) = R^2 + \alpha f_1(\varphi) + \ldots \tag{2.2}
$$

$$
\Phi(\zeta, \varphi) = \Phi_0 + \alpha\Phi_1 + \ldots =
$$

$$
\frac{F_0(\zeta, \varphi) + \alpha F_1(\zeta, \varphi) + \ldots}{\sqrt{\zeta^2 - \rho^2(\varphi)}} + \left(4A\zeta^2 - w_0 + \frac{8}{3}\alpha\zeta^2\cos 2\varphi\right) + \alpha G_1(\zeta, \varphi) + \ldots
$$

The contact pressure is hence determined from the formula

$$p(\rho, \varphi) = h\,\frac{F_0(\zeta, \varphi) + \alpha F_1(\zeta, \varphi) + \ldots}{\rho\sqrt{\rho^2(\varphi) - \rho^2}}, \quad \rho < \rho(\varphi) \tag{2.3}$$

Henceforth, we limit ourselves to an approximation of order $\alpha$. It is then required to determine the functions $F_0(\zeta, \varphi)$, $F_1(\zeta, \varphi)$ and the function $f(\varphi) = \alpha R^{-2} f_1(\varphi)$, which should have the form

$$f(\varphi) = c_0 + c_2 \cos 2\varphi + c_4 \cos 4\varphi + \ldots \tag{2.4}$$

because of the symmetry of the contact domain relative to the lines $\varphi = 0$ and $\varphi = \pi / 2$, in order to find the pressure $p(\rho, \varphi)$.

By using an expansion of the quantity $[\zeta^2 - R^2 - \alpha f_1(\varphi)]^{-1/2}$ in a series in $\alpha$ for $\alpha f_1(\varphi) < |\zeta^2 - R^2|$, we find from (2.2)

$$\Phi_0 = F_0 / \sqrt{\zeta^2 - R^2} + 4A\zeta - w_0$$

$$\Phi_1 = \frac{F_1(\zeta^2 - R^2) + {}^1/_2 f_1(\varphi) F_0}{(\zeta^2 - R^2)^{3/2}} + \frac{8}{3}\zeta^2 \cos 2\varphi + G_1$$

Requiring that the functions $\Phi_0$ and $\Phi_1$ vanish at infinity, and assure the regularity of the pressure in the neighborhood of the origin $\zeta = 0$, we obtain

$$F_0 = -4A\zeta^3 + (w_0 + 2AR^2)\,\zeta \tag{2.5}$$

$$F_1 = \frac{1}{2}\alpha^{-1}R^2\zeta(\zeta^2 - R^2)^{-1}\Big[-f(\varphi)F_0 + (w_0 - 2AR^2) \times \sum_{k=0}^{\infty} c_{2k}\left(\frac{\zeta}{R}\right)^{2k} \cos 2k\varphi\Big] - \frac{3}{8}\zeta^3 \cos 2\varphi$$

The expressions (2.3) and (2.5) show that the problem of finding the contact pressure reduces to constructing the trigonometric expansion (2.4) of the function $f(\varphi)$.

3. Let us use the condition on the boundary of the contact domain to determine the coefficients $c_{2k}$ in the expansion (2.4).

Evidently, on the circular part of the boundary

$$f(\varphi) = 0 \tag{3.1}$$

while on the remaining, previously unknown, part

$$p(\rho, \varphi)\,|_{\rho^2 = R^2[1 + f(\varphi)]} = 0 \tag{3.2}$$

To the accuracy of terms containing $\alpha^2$, the relationship (3.2) can be rewritten in conformity with (2.3) – (2.5) in the form ($\delta$ is the dimensionless displacement of the stamp)

$$f(\varphi) + D\sum_{k=0}^{\infty} c_{2k} k \cos 2k\varphi = -2D - \varepsilon \cos 2\varphi \tag{3.3}$$

$$\varepsilon = 3\alpha(16A)^{-1}, \quad D = {}^1/_4(2 - \delta), \quad \delta = w_0 (AR^2)^{-1}$$

The function $f(\varphi)$ characterizes the deviation of the contact domain outline from the circle $\rho^2 = R^2$. Therefore, it should be continuous and piecewise-monotonic in the interval $[0, 2\pi]$, and should also take on identical values at the points

$\varphi = 0$ and $\varphi = 2\pi$.

Since

$$f(\varphi) = f(2\pi - \varphi), \quad f(\pi / 2 + \varphi) = f(\pi / 2 - \varphi) \tag{3.4}$$

the requirement $f(0) = f(2\pi)$ is satisfied automatically, and the continuity of the function $f(\varphi)$ is assured by compliance with the condition $f(\psi - 0) = f(\psi + 0)$.

Formulas (3.1) and (3.3) form a system of relationships to find the coefficients of the expansion (2.4), where because of (3.4) it is sufficient that condition (3.1) hold in the interval $[0, \psi)$ and (3.3) in the interval $(\psi, \pi / 2]$.

It is characteristic that the position of the point $\psi$ is not fixed in advance, but should be determined in such a way that the constraints imposed on the function $f(\varphi)$ would be satisfied. In this connection, let us note the following. The function $f(\varphi)$ is absolutely continuous under compliance with the constraints mentioned [9]; consequently, the coefficients of the trigonometric series (2.4) tend to zero more rapidly than $1 / k$ [10]. Otherwise, it can only be asserted that $c_{2k} = o(1)$.

Therefore, the desired value of the parameter $\psi \in [0, \pi / 2]$ assures the most rapid decrease in the coefficients of the trigonometric expansion of the function $f(\varphi)$ governed by the relationships (3.1) and (3.3) for a given value of the constant $D$, or equivalently, the constant $\delta$.

In particular, if $\delta = 2$ $(D = 0)$, then conditions (3.1) and (3.3) permit the immediate conclusion that $\psi = \pi / 4$. Applying the formula to calculate the Fourier coefficients, we obtain

$$c_0 = \varepsilon / \pi, \quad c_2 = -{}^1/_2\varepsilon$$

$$c_{4k+6} = 0, \quad c_{4k+4} = (-1)^k [(2k+1)(2k+3)] \quad (k = 0, 1, 2, \ldots)$$

Therefore, the parameter $\psi$ and the Fourier coefficients of the function $f(\varphi)$ are determined exactly for $\delta = 2$.

The exact solution of the problem under consideration is found successfully in the limit cases also when a) only two diametrally opposite points of the endface line of the stamp are contiguous to the elastic medium in the $zx$ plane, and b) the whole line of the stamp endface, with the exception of two points in the $zy$ plane, makes contact with the half-space (passage to a circular contact zone).

In the first case, condition (3.3) is given in the whole interval $(0, \pi / 2]$, and compliance with the condition (3.1) is required at the point $\psi = 0$. We obtain from the relationship (3.3) whose right side is a Fourier polynomial

$$c_0 = -2D, \quad c_2 = -\varepsilon(1 + D), \quad c_{2k} = 0, \quad k > 1$$

Using (3.1), we determined the appropriate value of the constant

$$\delta_- = 2(2 - \sqrt{1 - 2\varepsilon})$$

In the second case, condition (3.3) is given at the point $\psi = \pi / 2$ while (3.1) is extended to the half-segment $[0, \pi / 2)$. This latter permits the conclusion that $c_{2k} \equiv 0$. We then have from (3.4)

$$\delta_+ = 2(1 - \varepsilon)$$

It is not possible to obtain the expansion (2.4) exactly for the remaining $\delta \in (\delta_-, \delta_+)$.

**4.** The reasoning presented in Sect. 3 relative to the criterion for determining the parameter $\psi$ permits the construction of an algorithm for the approximation solution of the problem by setting the collocation method [3, 11] as the basis for evaluating the coefficients of the trigonometric expansion of the function $f(\varphi)$.

Let us select the system of points

$$\delta_1 < \delta_2 < \ldots < \delta_m$$

in the interval $(\delta_-, \delta_+)$, and let us proceed as follows for each $\delta_j$ $(j = 1, 2, \ldots, m)$.

We put $c_{2k} = 0$ for $k > N$ in (2.4), i.e., we replace the series by a trigonometric polynomial. By using the formula for instance

$$\varphi_s = (s + {}^1/_2)\pi \,/\, (2N + 2), \quad s = 0, 1, \ldots, N \tag{4.1}$$

we give $N + 1$ equidistant points in the interval $(0, \pi / 2)$.

Requiring compliance with condition (3.1) at the points $0 < \varphi_s \leqslant \varphi_j$, and with (3.3) at the points $\varphi_{j+1} \leqslant \varphi_s < \pi / 2$, and giving $\varphi_j$ the values $\varphi_0, \varphi_1, \ldots, \varphi_N$ successively, we arrive each time at a system of linear algebraic equations

$$\begin{gathered}\sum_{k=0}^{N} c_{2k} \cos 2k\varphi_s = 0, \quad 0 < \varphi_s \leqslant \varphi_j \\ \sum_{k=0}^{N} c_{2k}(1 + Dk)\cos 2k\varphi_s = -2D - \varepsilon \cos 2\varphi_s, \quad \varphi_{j+1} \leqslant \varphi_s < \frac{\pi}{2}\end{gathered} \tag{4.2}$$

for the approximate determination of the coefficients $c_{2k}$ $(k = 0, 1, \ldots, N)$.

We select as the desired solution from those for the system (4.1) corresponding to the mentioned values of $\varphi_j$, that which forms the sequence of quantities $c_{2k}$ which decreases most rapidly in absolute value. The $\varphi_j$ corresponding to this solution thereby determines the interval $[\varphi_j, \varphi_{j+1}]$ in which the desired value of $\psi$ is contained. In particular, we set

$$\psi \approx {}^1/_2\,(\varphi_j + \varphi_{j+1}) \tag{4.3}$$

After finding the approximate values of the first $N + 1$ coefficients of the series (2.4) and determining the parameter $\psi$, the boundary of the contact domain can be constructed for given $\delta_j$ $(j = 1, 2, \ldots, m)$ by the first formula in (2.2), and the pressure distribution $p(\rho, \varphi)$ in the contact zone by the formulas (2.3) and (2.5).

The algorithm to solve the problem was written in the language ALGOL–60 and tested in the computer for different values of the ratio $\lambda = a / b$ governing the stamp geometry. Curves characterizing the dependence of the position of the point $\psi$ on the magnitude of the dimensionless displacements $\delta$ are presented in Fig. 2. The

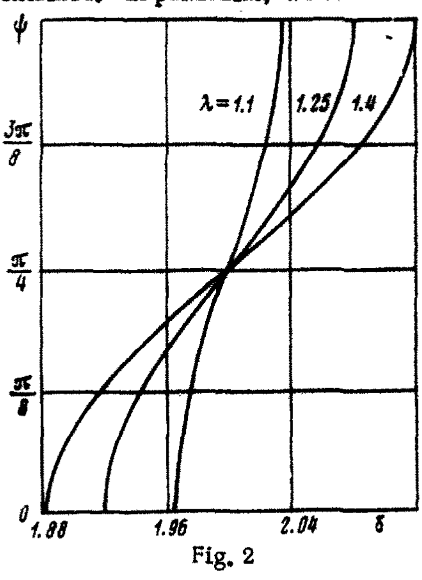

Fig. 2

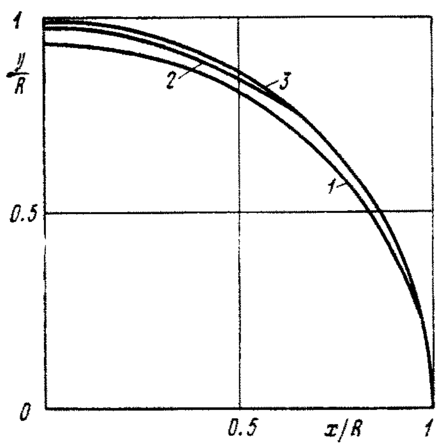


Fig. 3

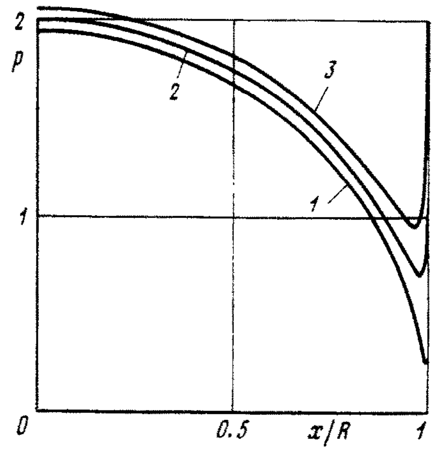


Fig. 4

calculations were performed for $N = 31$, $\delta = 2 \pm l\Delta$, $\Delta = 0.0025$, $l = 1, 2, \ldots, [(2 - \delta_-) / \Delta]$ in conformity with (4.1) and (4.3), where the construction of the curves for the values of $\delta$ close to $\delta_-$ and $\delta_+$ was refined by means of a quadruple subdivision of the step $\Delta$. The result obtained agrees with the physical representations of the nature of the stamp making contact with the half-space: elongation of the circular part of the contact domain boundary in conformity with the elongation of the sections of the endface line adjacent to the elastic medium occurs more intensively with the growth of $\delta$ near $\delta = \delta_-$ and $\delta = \delta_+$, i.e., during the passage from one kind of contact interaction to another. The position of the contact domain boundary for $\lambda = 1.4$ and $\delta = 1.89125, 2, 2.11125$ is shown in Fig. 3 by the curves *1, 2, 3*, respectively. The contact pressure distribution diagrams on the axis $\varphi = 0$, as constructed by means of (2.3) and (2.5) to the accuracy of the factor $2.4Rh$, and placed in Fig. 4, correspond to the same values of $\lambda$ and $\delta$.

## REFERENCES

1. Mossakovskii, V. I. The fundamental mixed problem the theory of elasticity for a half-space with a circular line of separation of the boundary condition, PMM, Vol. 18, No. 2, 1954.

2. Mossakovskii, V. I. The pressure of a punch of almost circular planform upon an elastic half-space, PMM, Vol. 18, No. 6, 1954.

3. Golubev, V. V., Lectures on Wing Theory, Gostekhizdat, Moscow —Leningrad, 1949.

4. Rvachev, V. L. and Protsenko, V. S., On the structure of the solution of a contact problem with an oblique stamp. Dopov. Akad. Nauk, UkrSSR, Ser. A, No. 11, 1970.

5. Rvachev, V. L., Protsenko, V. S., and Arkhipova, E. S., Structural method and its application to the solution of three-dimensional contact problems of elasticity theory. Tr. Conf.: Computational Mathematics in Modern Scientific-Technical Progress. Kiev, 1974.

6. R v a c h e v, V. L., Methods of Logic Algebra in Mathematical Physics, Naukova Dumka, Kiev, 1974.

7. G a l i n, L. A., Contact Problem of Elasticity Theory, Gostekhizdat, Moscow, 1953.

8. M o s s a k o v s k i i, V. I., Dependence between the force and settlement for an almost circular flat stamp, IN: Hydroaeromechanics and Elasticity Theory, No. 14, Izd. Dnepropetrovsk Univ., 1972.

9. N a t a n s o n, I. P., Theory of Functions of a Real Variable, "Nauka", Moscow, 1974.

10. K h a r d y, G. Kh. and R o g o z i n s k i, W. W., Fourier Series, Fizmatgiz, Moscow, 1962.

11. G a l i n, L. A., Solution of elasticity theory boundary value problems by the method of point interpolation, PMM, Vol. 3, No. 4, 1939.

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